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(Generalized) Boolean functions: invariance under some groups of transformations and differential properties

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Estimated success of brute force attacks for various key sizes

- 56 bits – 1 million-keys/sec (desktop PC) – 2,283 years
- 56 bits – 1 billion-keys/sec (medium corporate) – 2.3 years
- 56 bits – 100 billion-keys/sec (nations) – 8 days
- 128 bits – 1 bilion-keys/sec (medium corporate) – 10^{22} yrs
- 128 bits – 10^{18} keys/sec (large corp.) – $10,783$ billion yrs
- 128 bits – 10^{32} keys/sec (nations; quantum) – 108 million yrs
- 192 bits – 10^{9} keys/sec (medium corp.) – $2 \cdot 10^{41}$ years
- 192 bits – 10^{18} keys/sec (large corp.) – $2 \cdot 10^{32}$ years
- 192 bits – 10^{23} keys/sec (nations; quantum) – $2 \cdot 10^{27}$ yrs
- 256 bits – 10^{23} keys/sec (nations; quantum) – $3.7 \cdot 10^{46}$ yrs
- 256 bits – 10^{32} keys/sec (nations; quantum) – $3.7 \cdot 10^{37}$ yrs

The objects of the investigation: (Generalized) Boolean functions I

- Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$
- *Generalized Boolean function* $f : \mathbb{V}_n \to \mathbb{Z}_q$ $(q \geq 2)$; its set $\mathcal{GB}_n^q$; when $q = 2$, $\mathcal{B}_n$; $\mathbb{Z}_q$ is the ring of integers modulo q.
- If $2^{k-1} < q \leq 2^k$, for any $f \in \mathcal{GB}_n^q$ we associate a unique sequence of Boolean fcts. $a_i \in \mathcal{B}_n$ $(0 \leq i \leq k-1)$ s.t.

$$f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1}a_{k-1}(\mathbf{x}), \forall \mathbf{x} \in \mathbb{V}_n.$$

- For $f : \mathbb{V}_n \to \mathbb{Z}_q$ in $\mathcal{GB}_n^q$ we define the *generalized Walsh-Hadamard transform* to be the complex valued function

$$\mathcal{H}_f^{(q)}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{V}_n} \zeta_q^{f(\mathbf{x})} (-1)^{\langle \mathbf{u}, \mathbf{x} \rangle},$$

where $\zeta_q = e^{\frac{2\pi i}{q}}$ and $\langle \mathbf{u}, \mathbf{x} \rangle$ denotes a (nondegenerate) inner product on $\mathbb{V}_n$ (like $\mathbf{u} \cdot \mathbf{x}$ on $\mathbb{F}_2^n$, or $\mathrm{Tr}(ux)$ on $\mathbb{F}_{2^n}$);

The objects of the investigation: (Generalized) Boolean functions II

- For $q = 2$, we obtain the usual *Walsh-Hadamard transform*

$$\mathcal{W}_f(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{V}_n} (-1)^{f(\mathbf{x})}(-1)^{\langle \mathbf{u}, \mathbf{x} \rangle}.$$

- A function $f : \mathbb{V}_n \to \mathbb{Z}_q$ is called *generalized bent* (*gbent*) if $|\mathcal{H}_f(\mathbf{u})| = 2^{n/2}$ for all $\mathbf{u} \in \mathbb{V}_n$.
- It generalizes *bents* f for which $|\mathcal{W}_f(\mathbf{u})| = 2^{n/2}$, $\forall \mathbf{u} \in \mathbb{V}_n$; equivalently, $N_f = 2^{n-1} \pm 2^{\frac{n}{2}-1}$ (distance from the set of all affine functions). These only exists for even n.

Counting bents I

- Bents are hard to construct and/or count: $(2^{n/2})!\, 2^{2^{n/2}} \le \#\,\text{bent} \le 2^{2^{n-1}+\frac{1}{2}\binom{n}{n/2}}$ or the more complicated Carlet-Klapper (2002) bound
- Agievich (bent rectangles, '07); Climent et al. ('08,'14) iterative constructions; better bounds for $n = 12, 14$ but become worse for n larger;
- Natalia (Tokareva) "hypothesizes" that the lower bound might be: $2^{2^{n-2}+\frac{1}{4}\binom{n}{n/2}}$, or perhaps asymptotically,

$$\#\,\text{bent} \sim 2^{2^{n-c}+d\binom{n}{n/2}},$$

for some constants c, d, with $1 \le c \le 2$.

Counting bents II

n	lower bound	# bent	upper bound	# Boolean
2	8	8	8	16
4	384	896	2,048	65,536
6	$2^{23.3}$	$2^{32.3}$	2^{38}	2^{64}
8	$2^{95.6}$	$2^{106.291}$	$2^{129.2}$	2^{256}
10	$2^{262.16}$	?	2^{612}	2^{1024}

- Preneel (1990), Meng et al. (2006): $B_6 = 5425430528$
- Langevin et al. (Dec. 2007):
  $B_8 = 99270589265934370305785861242880 \sim 2^{106.291}$

Applications of (generalized) Boolean functions

- S-Boxes for block ciphers. e.g. DES, AES
- ’Combiners’ or ’filters’ for Linear Feedback Shift Registers (LFSRs) based stream ciphers: the ’Grain’ family of ciphers (eSTREAM project in Europe), Bluetooth E0, E1, etc.
- Coding theory; e.g. Reed-Muller code
- Spread spectrum communication; e.g., 4G-CDMA=3G-CDMA+OFDM; MC-CDMA=OFDM+CDMA, etc.
- In MC-CDMA systems, the symbol is spread by a user specific spreading sequence, and converted into a parallel data stream, which is then transmitted over multiple carriers.

Peak-to-Power Ratio – System Model I

- Let $n = 2^m$ and H_n be the canonical Walsh-Hadamard matrix of dimension 2^n; $\omega = \exp(2\pi\imath/2^h)$ be a primitive 2^h-th root of unity in $\mathbb{C}$, $h \in \mathbb{Z}^+$;
- Given a word $c = (c_1, \ldots, c_n)$, $c_i \in \mathbb{Z}_{2^h}$, the transmitted MC-CDMA signal can be modeled as

$$S_c(t) = \sum_{j=1}^{n-1} \omega^{c_j} (H_n)_{j,t}, 0 \le t < n,$$

(that is, c_j is used to modulate the j-th row of H_n, and the transmitted signal is the sum of these modulated sequences).

Peak-to-Power Ratio – System Model II

- The PAPR (peak-to-average-power ratio) of a codeword c (and code C) is defined by

$$PAPR(c) = \frac{1}{n} \max_{0 \le t < n} |S_c(t)|^2; \quad PAPR(C) = \max_{c \in C} PAPR(c).$$

The transmit signals in an orthogonal frequency-division multiplexing (OFDM) system can have high peak values in the time domain since many subcarrier components are added via an inverse fast Fourier transformation (IFFT) operation. As a result, OFDM systems are known to have a high peak-to-average power ratio (PAPR) when compared to single-carrier systems. In fact, the high PAPR is one of the most detrimental aspects in an OFDM system as it decreases the signal-to-quantization noise ratio (SQNR) of the analog-digital convertor (ADC) and digital-analog convertor (DAC) while degrading the efficiency of the power amplifier in the transmitter. As a side note, the PAPR problem is more of a concern in the uplink since the efficiency of the power amplifier is critical due to the limited battery power in a mobile terminal.

GOOGLE (Nutaq)

Peak-to-Power Ratio – System Model III

- A major problem to overcome: minimize peak-to-power ratio (PAPR);

Theorem (Schmidt (2009))

Let $f : \mathbb{F}_2^n \to \mathbb{Z}_{2^h}$ be a generalized Boolean function. Then,

$$PAPR(c) = \frac{1}{2^n} \max_{\mathbf{u} \in \mathbb{Z}_2^n} |\mathcal{H}_f^{(2^h)}(u)|^2.$$

In particular, the PAPR of f is 1 if and only if f is gbent.

Existence Results: from $\mathbb{F}_2^n \to \mathbb{Z}_{2^k}$ (the set $\mathcal{GB}_n^{2^k}$)

- Subsets of {S., Gangopadhyay, Martinsen, Singh, Meidl, Mesnager, Pott, Hodžić, Pasalic, Tang, Xiang, Qi, Feng}.: analyzed and constructed large classes of generalized bents; we now have a complete characterization of gbents in terms of their components.

Theorem (2016)

Let $f : \mathbb{F}_{2^n} \to \mathbb{Z}_{2^k}$, n even. Then f is a gbent function given as $f(x) = a_0(x) + 2a_1(x) + \cdots + 2^{k-1}a_{k-1}(x)$ if and only if, for each $\mathbf{c} \in \mathbb{F}_2^{k-1}$, the Boolean function $f_{\mathbf{c}}$ defined as

$$f_{\mathbf{c}}(x) = c_0 a_0(\mathbf{x}) \oplus c_1 a_1(x) \oplus \cdots \oplus c_{k-2} a_{k-2}(x) \oplus a_{k-1}(x)$$

is a bent function, such that $\mathcal{W}_{f_{\mathbf{c}}}(a) = (-1)^{\mathbf{c}\cdot g(a)+s(a)} 2^{\frac{n}{2}}$, for some $g : \mathbb{F}_{2^n} \to \mathbb{Z}_{2^{k-1}}$, $s : \mathbb{F}_{2^n} \to \mathbb{F}_2$.

Differential properties of generalized Boolean functions I

- $\mathbf{u} \in \mathbb{V}_n$ is a *linear structure* of $f \in \mathcal{GB}_n^q$ if the derivative of f wrt $\mathbf{u}$ is constant, that is, $f(\mathbf{x} \oplus \mathbf{u}) - f(\mathbf{x}) = c \in \mathbb{Z}_q$ constant, for all $\mathbf{x} \in \mathbb{V}_n$.
- Let $S_f = \{\mathbf{x} \in \mathbb{V}_n \,|\, \mathcal{H}_f(\mathbf{x}) \neq 0\} \neq \emptyset$ (gen.WH support)

Theorem (2017)

Let $f \in \mathcal{GB}_n^{2^k}$. Then a vector $\mathbf{u}$ is a linear structure for f iff $\zeta^{f(\mathbf{u})-f(\mathbf{0})} = (-1)^{\mathbf{u}\cdot\mathbf{w}}$, for all $\mathbf{w} \in S_f$. As a consequence, if $\mathbf{u}$ is a linear structure for f, then $f(\mathbf{u}) - f(\mathbf{0}) \in \{0, 2^{k-1}\}$.

Differential properties of generalized Boolean functions II

- **Corollary:** *Let $f \in \mathcal{GB}_n^{2^k}$. If $\mathbf{u}$ is a linear structure for f, then either $S_f \subseteq \mathbf{u}^\perp$, or $S_f \subseteq \overline{\mathbf{u}^\perp}$ (the set complement of $\mathbf{u}^\perp$).*

Theorem (2017)

Let $f \in \mathcal{GB}_n^{2^k}$, $k \geq 2$, be given by $f(\mathbf{x}) = \sum_{i=0}^{k-1} 2^i a_i(\mathbf{x})$, $a_i \in \mathcal{B}_n$. Then $\mathbf{u} \in \mathbb{V}_n$ is a linear structure for f iff $\mathbf{u}$ is a linear structure for a_i, $i \geq 0$, such that $a_i(\mathbf{u}) = a_i(\mathbf{0}), 0 \leq i < k-1$.

Differential properties of generalized Boolean functions III

- Using the method of Lechner ('71) and Lai ('95) one can simplify the ANF of a function admitting linear structures.

Theorem (2017)

Let $f \in \mathcal{GB}_n^{2^k}$ and $1 \leq \dim LS_{2^k}(f) = r$. Then, $\exists$ an invertible $n \times n$ matrix A such that

$$f((x_1, \ldots, x_n) \cdot A) = \sum_{i=1}^{r} \alpha_i x_i + g(x_{r+1}, \ldots, x_n),$$

where $\alpha_i \in \mathbb{Z}_{2^k}$ and $g \in \mathcal{GB}_{n-r}^{2^k}$ has no linear structures.

Differential properties of generalized Boolean functions IV

- We say that $f \in \mathcal{GB}_n^{2^k}$ satisfies the (generalized) *strict avalanche criterion* if the autocorrelation $\mathcal{C}_f(\mathbf{e}) = \sum_{\mathbf{x}\in\mathbb{V}_n} \zeta^{f(\mathbf{x})-f(\mathbf{x}\oplus\mathbf{e})} = 0$, for all $\mathbf{e}$ of weight 1.

Theorem (2017)

Let $f \in \mathcal{GB}_n^{2^k}$, and $A_j^{(\mathbf{w})} = \{\mathbf{x} | f(\mathbf{x}\oplus\mathbf{w}) - f(\mathbf{x}) = j\}$. Then f satisfies the SAC iff $|A_j^{(\mathbf{e})}| = |A_{j+2^{k-1}}^{(\mathbf{e})}|$, for all $0 \le j \le 2^{k-1}-1$, $wt(\mathbf{e}) = 1$. Also, f is gbent if and only if $|A_0^{(\mathbf{0})}| = 2^n, |A_j^{(\mathbf{0})}| = 0, |A_j^{(\mathbf{w})}| = |A_{j+2^{k-1}}^{(\mathbf{w})}|$, $0 \le j \le 2^{k-1}-1, \mathbf{w} \ne 0$.

Correlation Immune Functions I

- A generalized Boolean function $f \in \mathcal{GB}_n^q$ is said to be *correlation immune of order* t, $1 \leq t \leq n$ if for any fixed subset of t variables the probability that, given the value of $f(\mathbf{x})$, the t variables have any fixed set of values, is 2^{-t}.
- An $m \times n$ array $OA(m, n, s, t)$ with entries from a set of s elements is called an *orthogonal array of size m with n constraints, s levels, strength t, and index r*, if any set of t columns of the array contain all s^t possible row vectors exactly r times.

Correlation Immune Functions II

- As expected, there's a connection with orthogonal arrays;

Theorem (2017)

Every order t correlation immune generalized Boolean function, $f \in \mathcal{GB}_n^q$, "involves" a partition of $\mathbb{V}_n$, consisting of q binary orthogonal arrays, each of strength t.

- Nice connections and constructions of SAC, CI, dependent upon labeling of the hypercube are in (my student) Thor Martinsen's PhD thesis.

Correlation Immune Functions III

Table: A CI(1) Generalized Boolean Function, $f \in \mathcal{GB}_4^4$

$\mathbb{F}_2^4$	f
0000	0
0001	3
0010	2
0011	1
0100	1
0101	2
0110	3
0111	0
1000	2
1001	1
1010	0
1011	3
1100	3
1101	0
1110	1
1111	2

Trade-offs for generalized Boolean functions I

- Are there symmetric and gbent generalized Boolean functions ($k > 1$)?
- **Theorem (2017)**: NO! (proof based upon Savicky's symmetric bents and the recent work on gbents)
  ****
- What about balanced and symmetric generalized Boolean functions ($k > 1$)?
- **Theorem (2017)**: NO! (hard to show – dio. eq.)
- Recall $X(d, n) = \sum_{i_1 < i_2 < \cdots < i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$:

Theorem (Cusick-Li-S., 2009)

If t, ℓ are positive integers, then $X(2^t, 2^{t+1}\ell - 1)$ is balanced.

Trade-offs for generalized Boolean functions II

- We conjectured that these are the only balanced elementary symmetric (many cases covered, but still open);
- (Cusick-Li-S. 2009):
  - If $d = 2^t + 1$, $n = 2^{t+1}\ell$, then $wt(X(2^t + 1, 2^{t+1}\ell)) = 2^{n-2}$;
  - If $d = 2^t$, $X(d, n)$ is balanced iff $n = 2^{t+1}\ell - 1$, $t, \ell \in \mathbb{Z}^+$;
  - If $d = 2^{t+1}\ell + r - 1$, $t, \ell > 0$, $0 \le r \le 2^{t+1}$, $2^t < d \le 2^{t+1} - 2$ even, then $X(d, n)$ is not balanced;
- (Ou–Zhao 2012): Let $d = 2^{t+w}(2^{s+1} - 1), n = 2^{t+w+1}(2^{s+1} - 1) + 2^t q + m$, $m \in \{-1, 0\}$. Under some assumption on t, w, s, q, then $X(d, n)$ is not balanced.

Trade-offs for generalized Boolean functions III

- (Castro-Medina 2011) & (Guo-Gao-Zhao 2015): Conjecture 1 is true if n is large enough (dependent upon the degree), $n > -2\left(\log_2 \cos(\pi/2^r)\right)^{-1}$, where $2^{r-1} \le d < 2^r$. In particular, if d is not a power of 2, $X(d,n)$ is not balanced for large n.
- (Su-Tang-Pott 2013): If $d = 2^t$, Conjecture 2 holds in most cases, that is, $wt(X(d,n)) < 2^{n-1}$;
- (Gao-Liu- Zhang 2015): If $n = 2^{t+1}\ell - 1$, ℓ odd, $2^{t+1} \not| d$, $X(d,n)$ balanced iff $d = 2^k$, $1 \le k \le t$;
- (Castro-Gonzales-Medina 2015): More open cases are covered where Conjecture 1 holds.

Bisecting binomial coefficients I

- The existence of balanced elementary symmetric polynomials is related to the problem of bisecting binomial coefficients, that is, *solutions of*

$$\sum_{i=0}^{n} x_i \binom{n}{i} = 0, \quad x_i \in \{-1, 1\}. \tag{1}$$

- *Trivial Solutions*: Obviously, if n is even, then $\pm(1, -1, \ldots, -1, 1)$ are two solutions of (1). If n is odd, then $(\delta_0, \ldots, \delta_{\frac{n-1}{2}}, -\delta_{\frac{n-1}{2}-1}, \ldots, -\delta_0)$ are $2^{\frac{n+1}{2}}$ solutions of (1).

Research Question (Open for the past 25 years)

Find all nontrivial solutions of (1).

Bisecting binomial coefficients II

- There are sporadic cases when non-trivial solutions do appear: e.g., if $n \equiv 2 \pmod 6$, since $\binom{n}{(n+1)/3} = \binom{n}{(n+1)/3-1} + \binom{n}{n-((n+1)/3-1)}$, nontrivial solutions always appear.
- Apart from this, all that was known about the bisection of binomial coefficients was mostly computational.
- (Mitchell, 1990): found the nontrivial solutions for $n = 8, 13$;
- (Cusick & Li, 2005): found all solutions of (1) when $n \leq 28$; nontrivial solutions exist iff $n = 8, 13, 14, 20, 24, 26$.
- (Ionascu-Martinsen-S., 2017): found all nontrivial solutions for $n \leq 51$.

Our approach on the problem I

- The binomial coefficients bisection can be thought of as a subset sum problem. The view we take is the following: a binomial coefficients bisection $\sum_{i\in I}\binom{n}{i} = \sum_{i\in \bar{I}}\binom{n}{i}$ will generate a solution to the Boolean equation

  $$\sum_{i=0}^{n} x_i \binom{n}{i} = 2^{n-1}, x_i \in \{0,1\}$$

  by taking $x_i = 1$ for $i \in I$ and $x_i = 0$, for $i \in \bar{I}$. Certainly, the reciprocal is true, as well, and so, we have an equivalence between these two problems.

Our approach on the problem II

- In general, given a set of positive integers $A = \{a_1, \ldots, a_N\}$ and $b \leq \frac{1}{2}\sum_i a_i$, $b \in \mathbb{N}$, one investigates the Boolean equation
$$\sum_{i=1}^{N} x_i a_i = b,\ x_i \in \{0, 1\}.$$
- The advantage of our approach is that these equations were studied before by analytical number theory methods and much (well, some) is known.
- In general, these problem are well known to be NP-complete [Garey–Johnson, 1979] and have many applications in cryptography, such as the Merkle-Hellman cryptosystem (1978).

Our approach on the problem III

- The density of $\mathcal{S} = \{a_1, \ldots, a_N\}$ is $d(\mathcal{S}) = \dfrac{N}{\log_2\left(\max\limits_{1\le i\le N} a_i\right)}$;

  in terms of knapsack cryptosystems,

  $d(\mathcal{S}) = \dfrac{\text{bit size of the plaintext}}{\text{average bit size of the ciphertext}}$

- For $\mathbf{P}_n = \left\{ \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \right\}$, using $\frac{4^{\lfloor n/2 \rfloor}}{2\lfloor n/2\rfloor+1} \le \binom{n}{\lfloor n/2 \rfloor} \le 4^{\lfloor n/2 \rfloor}$, the density becomes

$$\frac{n+1}{2\lfloor n/2\rfloor - \log_2(2\lfloor n/2\rfloor + 1)} \le d(\mathbf{P}_n) = \frac{n+1}{\log_2(\max_i \binom{n}{i})} = \frac{n+1}{\log_2 \binom{n}{\lfloor n/2\rfloor}} \le \frac{n+1}{2\lfloor n/2\rfloor},$$

and so,

$$d(\mathbf{P}_n) \to 1, \text{ as } n \to \infty.$$

Our approach on the problem IV

- Lagarias and Odlyzko (1985) showed that almost all the subset sum problem with density $d < 0.6463\ldots$ can be solved in polynomial time with a single call to an oracle that can find (in polynomial time with high probability) the shortest vector in a special lattice. Coster et al. (1992) improved the bound to $d < 0.9408\ldots$.
- Since for binomial coefficients, the density is $d = 1$ (as $n \to \infty$), none of these methods are applicable.

The underlying method I

- We recall here the following important result of Freiman (1980) (see also [Buzytsky (1982), Chaimovich, Freiman, Galil (1989)]).

Theorem (Freiman)

Let $A = \{a_1, a_2, \ldots, a_N\}$ and $b \leq \frac{1}{2}\sum_{i=1}^{N} a_i$. The number of Boolean solutions for the equation

$$\sum_{i=1}^{N} a_i x_i = b, \;\; x_i \in \{0, 1\}$$

is precisely $\displaystyle\int_0^1 e^{-2\pi i x b} \prod_{j=1}^{N} \left(1 + e^{2\pi i x a_j}\right) d\,x.$

The underlying method II

- Applying Freiman's paradigm to the bisection of bin. coeff.:

Theorem (Ionascu-Martinsen-S., 2017)

The number of binomial coefficients bisections for fixed n is exactly

$$J_n = \int_0^1 e^{-2^n \pi i x} \prod_{j=0}^{n} \left(1 + e^{2\pi i x \binom{n}{j}}\right) d\,x = 2^{n+1} \int_0^1 \prod_{j=0}^{n} \cos\left(\pi x \binom{n}{j}\right) d\,x.$$

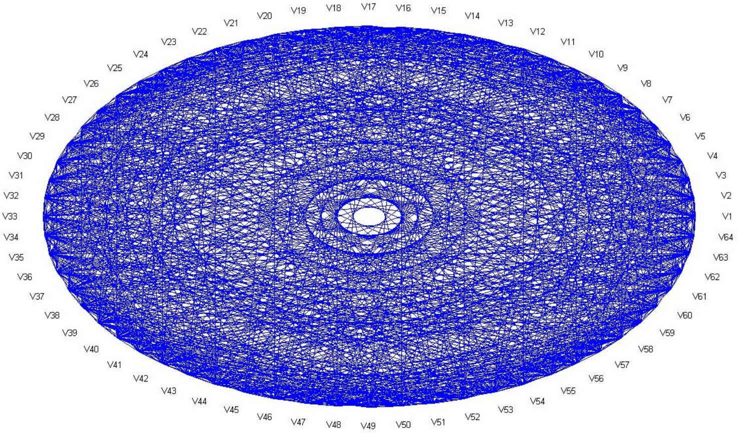
- We constructed infinite families with nontrivial, as well as infinite families with only trivial bisections.
- As a by-product, we got for free two conjectures of Cusick et al. ('05), so there are only four symmetric SAC(k) functions for infinitely many n.

Go2Graphs Go2Tks

Visualizing Boolean functions

- Can one visualize Boolean functions?
- Yes, in several ways, but it becomes very hard to obtain results just based upon graph theoretical tools.
- Nagy graphs, Cayley graphs, etc.
- E.g.: (undirected) Cayley graph – vertices are points of $\mathbb{F}_2^n$ and two points $\mathbf{x}, \mathbf{y}$ are connected by an edge iff $f(\mathbf{x} \oplus \mathbf{y}) = 1$.

Cayley graph of first row of S-box 1 of DES



Go2Tks

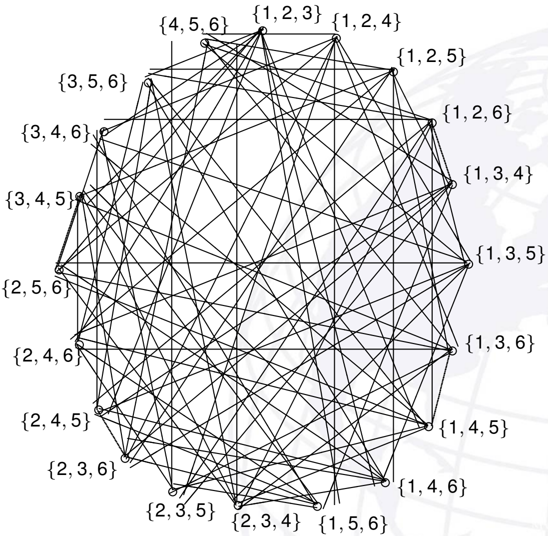
Further Restrictions: invariance under a group of transformations

- On $\mathbb{F}_2^6$, there are 2^{20} cubic homogeneous B.f.
- Among these, $\exists$ 30 homogeneous bent B.f. equivalent to Rothaus ('76): $x_1x_2x_3 \oplus x_1x_4 \oplus x_2x_5 \oplus x_3x_6$
- Qu-Seberry-Pieprzyk (2000): *There are* $> 30^n\binom{6n}{6}$ *homogeneous bent B.f. on* $\mathbb{F}_2^{6n}$.
- Charnes-Rötteler-Beth (2002):
  The bent functions found by Qu et al. ('00) arise as invariants under the action of the symmetric group on four letters;

Definition (Nagy Graph)

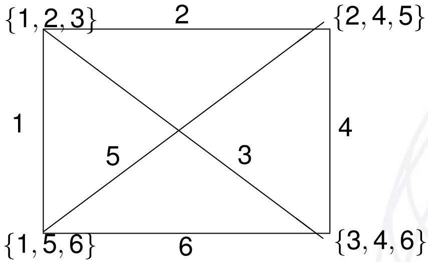
$\Gamma_{(n,k)}$: vertices – the $\binom{n}{k}$ unordered subsets of size k of $\{1,\ldots,n\}$; vertices are joined by an edge whenever the corresponding k-sets intersect in a subset of size one.

Nagy graph $\Gamma_{(6,3)}$



Cliques and Homogeneous Bent Functions

- A *clique* in an undirected graph Γ is a complete subgraph (maximal: not contained in a bigger one); *the clique problem*) is NP-complete.



ptmmn.

Theorem (Charnes-Rötteler-Beth (2002))

The thirty homogeneous bent functions in six variables listed by Qu et al. are in one to one correspondence with the complements of the 30 *(maximal) cliques of* $\Gamma_{(6,3)}$.

Open questions

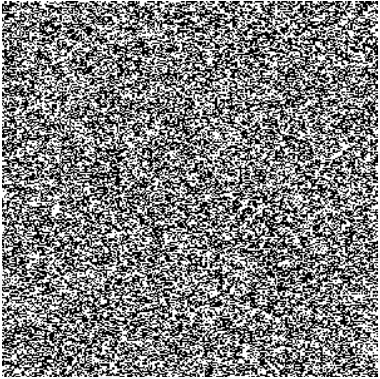
- It is unknown whether there are quartic/quintic/etc. homogeneous bent functions.
- I propose to look at the complements of the maximal cliques of the Nagy graph $\Gamma_{(10,4)}, \Gamma_{(12,4)}$.
- Do the same for $\Gamma_{(12,5)}, \Gamma_{(14,5)}$.

Research Question

Can one find efficiently a (all) clique(s) in $\Gamma_{(2n,k)}$, $k < n$?

- Not a trivial matter, I believe: for instance, $\Gamma_{(10,4)}$ has 210 vertices; $\Gamma_{(12,5)}$ has 792 vertices;

Having some fun: using a gen. Boolean as a combiner



Theorem (Pante Stanica: http://faculty/nps.edu/pstanica)

Thank you for your attention!

Proof.

None required! $\square$